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Colouring H -free Graphs of Bounded Diameter

Barnaby Martin

Department of Computer Science, Durham University, United Kingdom
barnaby.d.martin@durham.ac.uk

Daniël Paulusma

Department of Computer Science, Durham University, United Kingdom
daniel.paulusma@durham.ac.uk

Siani Smith

Department of Computer Science, Durham University, United Kingdom
siani.smith@durham.ac.uk

Abstract

The COLOURING problem is to decide if the vertices of a graph can be coloured with at most k colours for an integer k , such that no two adjacent vertices are coloured alike. A graph G is H -free if G does not contain H as an induced subgraph. It is known that COLOURING is NP-complete for H -free graphs if H contains a cycle or claw, even for fixed $k \geq 3$. We examine to what extent the situation may change if in addition the input graph has bounded diameter.

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1 Introduction

Graph colouring is one of the best studied concepts in Computer Science and Mathematics. This is mainly due to its many practical and theoretical applications and its many natural variants and generalizations. Over the years, numerous surveys and books on graph colouring were published (see, for example, [1, 4, 18, 21, 26, 28, 31]).

A (*vertex*) *colouring* of a graph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, 2, \dots\}$ that assigns each vertex $u \in V$ a *colour* $c(u)$ in such a way that $c(u) \neq c(v)$ whenever $uv \in E$. If $1 \leq c(u) \leq k$, then c is said to be a k -*colouring* of G and G is said to be k -*colourable*. The COLOURING problem is to decide if a given graph G has a k -colouring for some given integer k . If k is *fixed*, that is, k is not part of the input, we denote the problem by k -COLOURING. It is well known that even 3-COLOURING is NP-complete [23].

In this paper we aim to increase our understanding of the computational hardness of COLOURING. One way to do this is to consider inputs from families of graphs to learn more about the kind of graph structure that causes the hardness. This led to a highly extensive study of COLOURING and k -COLOURING for many special graph classes. The best-known result in this direction is due to Grötschel, Lovász, and Schrijver, who proved that COLOURING is polynomial-time solvable for perfect graphs [13].

Perfect graphs form an example of a graph class that is closed under vertex deletion. Such graph classes are also called *hereditary*. Hereditary graph classes are ideally suited for a *systematic* study in the computational complexity of graph problems. Not only do they capture a very large collection of many well-studied graph classes, but they are also exactly the graph classes that can be characterized by a unique set \mathcal{H} of minimal forbidden induced subgraphs. When solving an NP-hard problem under input restrictions, it is standard



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practice to consider, for example, first the case where \mathcal{H} has small size, or where each $H \in \mathcal{H}$ has small size.

We note that the set \mathcal{H} defined above may be infinite. If not, say $\mathcal{H} = \{H_1, \dots, H_p\}$ for some positive integer p , then the corresponding hereditary graph class \mathcal{G} is said to be *finitely defined*. Formally, a graph G is (H_1, \dots, H_p) -free if for each $i \in \{1, \dots, p\}$, G is H_i -free, where the latter means that G does not contain an induced subgraph isomorphic to H_i .

We emphasize that the borderline between NP-hardness and tractability is often far from clear beforehand and jumps in computational complexity can be extreme. In order to illustrate this behaviour of graph problems, we present the following example of a (somewhat artificial) graph problem related to vertex colouring.

COLOURING-OR-SUBGRAPH

Instance: an n -vertex graph G

Question: is G $\lceil \sqrt{\log n} \rceil$ -colourable or H -free for some graph H with $|V(H)| \leq \lceil \sqrt{\log n} \rceil$?

► **Theorem 1.** *The COLOURING-OR-SUBGRAPH problem is NP-hard, but constant-time solvable for every hereditary graph class not equal to the class of all graphs.*

Proof. We reduce from 3-COLOURING, which we recall is NP-complete [23]. Let G be an n -vertex graph. Set $k = \lceil \sqrt{\log n} \rceil$. Add $k - 3$ pairwise adjacent vertices to G . Make the new vertices also adjacent to every vertex of G . Add each possible graph on k vertices as a connected component to G . The resulting graph G' has $n + (k - 3) + k \cdot 2^{\frac{k(k-1)}{2}} < 3n^2$ vertices. By construction, G' contains every graph on at most k vertices as an induced subgraph. Hence, G' is a yes-instance of COLOURING-OR-SUBGRAPH if and only if G' is k -colourable, and the latter holds if and only if G is 3-colourable.

Now let \mathcal{G} be a hereditary graph class for which there exist at least one graph H such that every graph $G \in \mathcal{G}$ is H -free. Let $\ell = |V(H)|$. We claim that COLOURING-OR-SUBGRAPH is constant-time solvable for \mathcal{G} . Let $G \in \mathcal{G}$ be an n -vertex graph. If $n \leq 2^{\ell^2}$, then G has constant size and the problem is constant-time solvable. If $n > 2^{\ell^2}$, then $\ell = |V(H)| < \sqrt{\log n} \leq \lceil \sqrt{\log n} \rceil$. Hence G is a yes-instance of COLOURING-OR-SUBGRAPH, as G is H -free and H has less than $\lceil \sqrt{\log n} \rceil$ vertices. ◀

In this paper, we consider the problems COLOURING and k -COLOURING. In order to describe known results and our new results we first give some terminology and notation.

1.1 Terminology and Notation

The *disjoint union* of two vertex-disjoint graphs F and G is the graph $G + F = (V(F) \cup V(G), E(F) \cup E(G))$. The disjoint union of s copies of a graph G is denoted sG . A *linear forest* is the disjoint union of paths. The *length* of a path or a cycle is the number of its edges. The *distance* $\text{dist}(u, v)$ between two vertices u, v in a graph G is the length of a shortest induced path between them. The *diameter* of a graph G is the maximum distance over all pairs of vertices in G . The *girth* of a graph G is the length of a shortest induced cycle of G . The graphs C_r , P_r and K_r denote the cycle, path and complete graph on r vertices, respectively.

A *polyad* is a tree where exactly one vertex has degree at least 3. We will use several special polyads in our paper. The graph $K_{1,r}$ denotes the $(r + 1)$ -vertex *star*, that is, the graph with vertices x, y_1, \dots, y_r and edges xy_i for $i = 1, \dots, r$. The graph $K_{1,3}$ is also called the *claw*. The *subdivision* of an edge uw in a graph removes uw and replaces it with a new vertex v and edges uv, vw . We let $K_{1,r}^\ell$ denote the ℓ -subdivided star, which is the graph

86 obtained from a star $K_{1,r}$ by subdividing one edge of $K_{1,r}$ exactly ℓ times. The graph $S_{h,i,j}$,
 87 for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, which is the tree with one vertex x of degree 3
 88 and exactly three leaves, which are of distance h , i and j from x , respectively. Note that
 89 $S_{1,1,1} = K_{1,3}$. The graph $S_{1,1,2} = K_{1,3}^1$ is also known as the *chair*.

90 1.2 Known Results

91 The computational complexity of COLOURING has been fully classified for H -free graphs:
 92 if H is an induced subgraph of $P_1 + P_3$ or of P_4 , then COLOURING for H -free graphs is
 93 polynomial-time solvable, and otherwise it is NP-complete [20]. In contrast, the complexity
 94 classification for k -COLOURING restricted to H -free graphs is still incomplete. It is known that
 95 for every $k \geq 3$, k -COLOURING for H -free graphs is NP-complete if H contains a cycle [10]
 96 or an induced claw [16, 22]. However, the remaining case where H is a linear forest has not
 97 been settled yet even if H consists of a single path. For P_t -free graphs, the cases $k \leq 2$, $t \geq 1$
 98 (trivial), $k \geq 3$, $t \leq 5$ [14], $k = 3$, $6 \leq t \leq 7$ [2] and $k = 4$, $t = 6$ [6] are polynomial-time
 99 solvable and the cases $k = 4$, $t \geq 7$ [17] and $k \geq 5$, $t \geq 6$ [17] are NP-complete. The cases
 100 where $k = 3$ and $t \geq 8$ are still open. For further details, including for linear forests H of more
 101 than one connected component, see the survey paper [11] or some recent papers [5, 12, 19].

102 1.3 Our Focus

103 We consider H -free graphs where H contains a cycle or claw. In this case, k -COLOURING
 104 restricted to H -free graphs is NP-complete for every $k \geq 3$, as mentioned above. However,
 105 we re-examine the situation after adding a diameter constraint to our input graphs. If the
 106 diameter is 1, then G is a complete graph, and COLOURING becomes trivial. As such, our
 107 research question is:

108 *To what extent does bounding the diameter help making COLOURING and k -COLOURING*
 109 *tractable on H -free graphs?*

110 We remark that H -free graphs of diameter at most d for some integer d are no longer
 111 hereditary, which requires some care in the proof of our results. We also note that by
 112 a straightforward reduction from 3-COLOURING one can show that k -COLOURING is NP-
 113 complete for graphs of diameter d for all pairs (k, d) with $k \geq 3$ and $d \geq 2$ except for two
 114 cases, namely $(k, d) \in \{(3, 2), (3, 3)\}$. Mertzios and Spirakis [24] settled the case $(k, d) = (3, 3)$
 115 by proving that 3-COLOURING is NP-complete even for C_3 -free graphs of diameter 3. The
 116 case $(k, d) = (3, 2)$ is still open.

117 1.4 Our Results

118 We complement the bounded diameter results of Mertzios and Spirakis [24] by presenting a
 119 set of new results for COLOURING and k -COLOURING for H -free graphs of bounded diameter
 120 when H contains a claw or a cycle. Results for the case where H has a cycle usually follow
 121 from stronger results for graphs of girth at least g for some fixed integer g . In particular,
 122 Emden-Weinert, Hougardy and Kreuter [10] proved that for all integers $k \geq 3$ and $g \geq 3$,
 123 k -COLOURING is NP-complete for graphs with girth at least g and with maximum degree at
 124 most $6k^{13}$ (for more results on COLOURING for graphs of maximum degree, see [3, 7, 25]).

125 First, in Section 3 we research the effect on bounding the diameter of k -COLOURING and
 126 COLOURING restricted to polyad-free graphs for various polyads. Our first result, which
 127 formed together with the result of [24] the starting point of our investigation, is that k -
 128 COLOURING is constant-time solvable for $K_{1,r}$ -free graphs of diameter d for any fixed integers

Colours	Diameter	H -free	Complexity	Theorem
fixed k	d	$K_{1,r}$	P	9
input k	d	$K_{1,4}$	NP-c	10
3	d	$K_{1,3}^1$	P	12(1)
3	2	$K_{1,r}^2$	P	12(2)
3	4	$K_{1,4}^3$	NP-c	12(3)
4	2	$K_{1,3}^1$	NP-c	12(4)
3	2	$S_{1,2,2}$	P	13

■ **Figure 1** Our polynomial-time (P) and NP-complete (NP-c) results for polyad-free graphs.

129 $d \geq 1$, $k \geq 1$ and $r \geq 1$. We also show that this does not hold for COLOURING (when k is
 130 part of the input). We then extend these results for larger polyads; see also Figure 1.

131 Second, in Section 4 we perform a similar study for graphs of bounded diameter and girth.
 132 We provide new polynomial-time and NP-hardness results for k -COLOURING, identifying and
 133 narrowing the gap between tractability and intractability, in particular for the case where
 134 $k = 3$ (see also Figure 2). Section 5 contains some open questions and directions for future
 135 work.

diameter \ girth	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	≥ 11	≥ 12
≤ 1	P	P	P	P	P	P	P	P	P	P
≤ 2	?	?	P	P	P	P	P	P	P	P
≤ 3	NP-c	NP-c	?	?	P	P	P	P	P	P
≤ 4	NP-c	NP-c	NP-c	NP-c	?	?	P	P	P	P
≤ 5	NP-c	NP-c	NP-c	NP-c	?	?	?	?	?	P

■ **Figure 2** The complexity of 3-COLOURING for graphs of diameter at most d and girth at least g .

136

137 2 Preliminaries

138 In this section we complement Section 1.1 by giving some additional terminology and notation.
 139 We also recall some useful results from the literature.

140 Let $G = (V, E)$ be a graph. A vertex $u \in V$ is *dominating* if u is adjacent to every other
 141 vertex of G . For a set $S \subseteq V$, the graph $G[S]$ denotes the subgraph of G induced by S . The
 142 *neighbourhood* of a vertex $u \in V$ is the set $N(u) = \{v \mid uv \in E\}$ and the *degree* of u is the size
 143 of $N(u)$. For a set $U \subseteq V$, we write $N(U) = \bigcup_{u \in U} N(u) \setminus U$. For a set $U \subseteq V$ and a vertex
 144 $u \in U$, the *private neighbourhood* of u with respect to U is the set $N(u) \setminus (N(U \setminus \{u\}) \cup U)$
 145 of *private neighbours* of u with respect to U , which is the set of neighbours of u outside U
 146 that are not a neighbour of any other vertex of U . If every vertex of G has degree p , then G
 147 is (p) -*regular*.

148 We will use the aforementioned results of Král' et al.; Holyer; Leven and Galil; Emden-
 149 Weinert, Hougardy and Kreuter; and Mertzios and Spirakis.

150 ► **Theorem 2 ([20]).** *Let H be a graph. If $H \subseteq_i P_4$ or $H \subseteq_i P_1 + P_3$, then COLOURING*
 151 *restricted to H -free graphs is polynomial-time solvable, otherwise it is NP-complete.*

152 ► **Theorem 3** ([16, 22]). *For every integer $k \geq 3$, k -COLOURING is NP-complete for claw-free*
 153 *graphs.*

154 ► **Theorem 4** ([10]). *For all integers $k \geq 3$ and $g \geq 3$, k -COLOURING is NP-complete for*
 155 *graphs with girth at least g (and with maximum degree at most $6k^{13}$).*

156 ► **Theorem 5** ([24]). *3-COLOURING is NP-complete for C_3 -free graphs of diameter 3.*

157 A *list assignment* of a graph $G = (V, E)$ is a function L that prescribes a *list of admissible*
 158 *colours* $L(u) \subseteq \{1, 2, \dots\}$ to each $u \in V$. A colouring c *respects* L if $c(u) \in L(u)$ for every
 159 $u \in V$. If $|L(u)| \leq 2$ for each $u \in V$, then L is also called a *2-list assignment*. The 2-LIST
 160 COLOURING problem is to decide if a graph G with a 2-list assignment L has a colouring
 161 that respects G . Our strategy for obtaining a polynomial-time algorithm for 3-COLOURING
 162 is often to reduce the input to a polynomial number of instances of 2-LIST COLOURING. The
 163 reason is that we can then apply the following well-known result of Edwards.

164 ► **Theorem 6** ([9]). *The 2-LIST COLOURING problem is linear-time solvable.*

165 We will also use the following result, which includes the Hoffman-Singleton Theorem,
 166 which provides a description of regular graphs of diameter 2 and girth 5.

167 ► **Theorem 7** ([8, 15, 30]). *For every $d \geq 1$, every graph of diameter d and girth $2d + 1$ is*
 168 *p -regular for some integer p . Moreover, if $d = 2$, then there are only four such graphs (with*
 169 *$p = 2, 3, 7, 57$, respectively) and if $d \geq 3$, then such graphs are cycles (of length $2d + 1$).*

170 A *clique* in a graph is a set of pairwise adjacent vertices, and an *independent set* is a set
 171 of pairwise non-adjacent vertices. By Ramsey's Theorem [27], there exists a constant, which
 172 we denote by $R(k, r)$, such that any graph on at least $R(k, r)$ vertices contains either a clique
 173 of size k or an independent set of size r .

174 **3 Polyad-Free Graphs of Bounded Diameter**

175 In this section we prove, among other things, our results on COLOURING and k -COLOURING
 176 for polyad-free graphs of bounded diameter; see also Figure 1. We first make an observation.

177 ► **Lemma 8.** *If G is a graph of diameter d that is not a tree, then G contains an induced*
 178 *cycle of length at most $2d + 1$.*

179 **Proof.** As G is not a tree and G is connected, G must contain a cycle C . Suppose that C has
 180 length at least $2d + 2$. Since G has diameter d , there exists a path of length at most d in G
 181 between any two vertices u and v at distance $d + 1$ in C . The vertices of this path, together
 182 with the vertices of the path of length $d + 1$ between u and v on C , induce a subgraph of G
 183 that contains an induced cycle C' of length at most $2d + 1$. ◀

184 We now state our first result, which forms the starting point of the research in this section.

185 ► **Theorem 9.** *For all integers $d, k, r \geq 1$, k -COLOURING is constant-time solvable for*
 186 *$K_{1,r}$ -free graphs of diameter d .*

187 **Proof.** Let $G = (V, E)$ be a $K_{1,r}$ -free graph of diameter d . We prove that if G has size
 188 larger than some constant $\beta(k, r)$, which we determine below, then G is not k -colourable. If
 189 $|V(G)| \leq \beta(k, r)$, we can solve k -COLOURING in constant time.

190 As G is $K_{1,r}$ -free, Ramsey's Theorem tells us that the neighbourhood of every vertex $u \in V$
 191 with degree at least $R(k, r)$ contains a clique of size k . In that case $N(u) \cup \{u\}$ is a clique of
 192 size $k + 1$. Hence, to be k -colourable, every vertex of G must have degree less than $R(k, r)$,
 193 so G must have at most $\beta(k, r) = 1 + R(k, r) + R(k, r)^2 + \dots + R(k, r)^d$ vertices. ◀

If k is not part of the input, Theorem 9 no longer holds. This is shown by the following more general theorem. In this theorem we assume that $H \not\subseteq_i P_1 + P_3$ and $H \not\subseteq_i P_4$, as in those cases COLOURING is polynomial-time solvable for all H -free graphs due to Theorem 2. Note that Theorem 10 covers all remaining cases except the case where $H = K_{1,3}$.

► **Theorem 10.** *Let H be a graph with $H \not\subseteq_i P_1 + P_3$ and $H \not\subseteq_i P_4$ and d be an integer. Then COLOURING for H -free graphs of diameter at most d is*

1. NP-complete if H has no dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$ and $d \geq 2$;
2. NP-complete if $H \neq K_{1,3}$ and H has a dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$ and $d \geq 3$.

Proof. 1. Let H have no dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$. We define H' as $H - u$ if H has a dominating vertex u and as H itself otherwise. By construction, $H' \not\subseteq_i P_1 + P_3$ and $H' \not\subseteq_i P_4$. Hence, COLOURING is NP-complete for H' -free graphs due to Theorem 2. Let G be an H' -free graph. Add a dominating vertex to G . The new graph G' has diameter 2 and is H -free. Moreover, G is k -colourable if and only if G' is $(k + 1)$ -colourable.

2. Let $H \neq K_{1,3}$ have a dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$. Then H cannot be a forest, as in that case H would be in $\{P_1, P_2, P_3, K_{1,3}\}$. Hence, H has an induced cycle C_r for some $r \geq 3$. If $r = 3$, then 3-COLOURING is NP-complete for H -free graphs of diameter 3, as it is so for C_3 -free graphs of diameter 3 due to Theorem 5. If $r \geq 4$, then COLOURING is NP-complete even for H -free graphs of diameter 2, as it is so for C_r -free graphs of diameter 2 due to 1. ◀

It is a natural question whether we can extend Theorem 9 to H -free graphs of diameter d , where H is a slightly larger tree than a star. The first interesting case is where H is an ℓ -subdivided star $K_{1,r}^\ell$ for some integer $\ell \geq 1$ and $r \geq 3$. We prove a number of results for various values of d, k, ℓ . For one of our proofs and also for the proof of our next result we need the following theorem.

► **Theorem 11.** *3-COLOURING can be solved in polynomial time for C_5 -free graphs of diameter at most 2.*

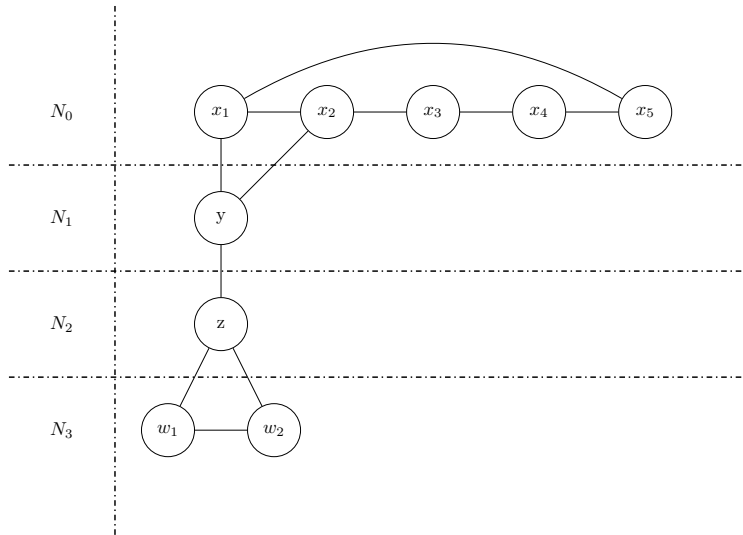
Proof. If G is bipartite, then G is 3-colourable. If G contains a K_4 , then G is not 3-colourable. We check these properties in polynomial time, and from now on we assume that G is K_4 -free and non-bipartite. The latter implies that G must have an odd induced cycle C_r for some odd integer r . As G has diameter 2, we find that $r \leq 5$ due to Lemma 8. As G is C_5 -free, it follows that $r = 3$.

Let C be a triangle in G . We write $N_0 = V(C) = \{x_1, x_2, x_3\}$, $N_1 = N(V(C))$ and $N_2 = V(G) \setminus (N_0 \cup N_1)$. As G has diameter 2, for every $i \in \{1, 2, 3\}$, it holds that every vertex in N_2 has a neighbour in N_1 that is adjacent to x_i .

We let T consist of all vertices of N_2 that have a neighbour in N_1 that is adjacent to exactly two vertices of N_0 . We claim that $N_2 = T$. In order to see this, let $u \in N_2$. If u has a neighbour $y \in N_1$ adjacent to every x_i , then G contains a K_4 , a contradiction. Hence, u must have three distinct neighbours y_1, y_2, y_3 , such that for $i \in \{1, 2, 3\}$, it holds that $N(y_i) \cap N_0 = \{x_i\}$. If $\{y_1, y_2, y_3\}$ is a clique, then G has a K_4 on vertices u, y_1, y_2, y_3 , a contradiction. Hence, we may assume without loss of generality that y_1 and y_2 are non-adjacent. However, then $\{u, y_1, x_1, x_2, y_2\}$ induces a C_5 in G , another contradiction. We conclude that $T = N_2$.

239 If G has a 3-colouring c , then we may assume without loss of generality that $c(x_i) = i$
 240 for $i \in \{1, 2, 3\}$. Hence, our algorithm assigns colours 1, 2, 3 to x_1, x_2, x_3 , respectively.
 241 This reduces the list of admissible colours of the vertices of N_1 by at least one colour. In
 242 particular, vertices in N_1 that have two neighbours in N_0 can be coloured with only one
 243 colour. Our algorithm assigns this colour to such vertices. This means that any of their
 244 neighbours in $T = N_2$ can be coloured with at most two colours. So, after propagation, we
 245 have obtained either two adjacent vertices that are coloured alike, in which case G is not
 246 3-colourable, or we have constructed an instance of 2-LIST COLOURING. We can solve such
 247 an instance in linear time due to Theorem 6. ◀

248 We are now ready to state our results for $K_{1,r}^\ell$, where we exclude the cases that are
 249 tractable in general, namely where $d = 1$, or $k \leq 2$, or $r \leq 2$ (the latter case corresponds to
 250 the case where $H = K_{1,2}^+ = P_4$, so we can use Theorem 2). Note that for $k \geq 4$ all interesting
 251 cases are NP-complete, whereas for $k = 3$ the situation is less clear.



■ **Figure 3** An example of a decomposition of a chair-free graph of diameter 3 into sets N_0, \dots, N_3 where $p = 5$ and $y \in N_1$ has two “descendants” in N_3 . To prevent an induced chair, y must be adjacent to exactly two (adjacent) vertices of N_0 , and w_1 and w_2 must be adjacent to each other.

252 ▶ **Theorem 12.** Let d, k, ℓ, r be four integers with $d \geq 2$, $k \geq 3$, $\ell \geq 1$ and $r \geq 3$. Then
 253 k -COLOURING for $K_{1,r}^\ell$ -free graphs of diameter at most d is:

- 254 1. polynomial-time solvable if $d \geq 2$, $k = 3$, $\ell = 1$ and $r = 3$
- 255 2. polynomial-time solvable if $d = 2$, $k = 3$, $\ell = 2$ and $r \geq 3$
- 256 3. NP-complete if $d \geq 4$, $k = 3$, $\ell \geq 3$ and $r \geq 4$
- 257 4. NP-complete if $d \geq 2$, $k \geq 4$, $\ell \geq 1$ and $r \geq 3$.

258 **Proof. 1.** Recall that $K_{1,3}^1$ is the chair $S_{1,1,2}$. Let G be a chair-free graph of diameter d .
 259 If G is a tree, then G is even 2-colourable. We check in $O(n^4)$ time if G has a K_4 . If so,
 260 then G is not 3-colourable. From now on we assume that G is not a tree and that G is
 261 K_4 -free. As G is not a tree and G is connected, G contains an induced cycle of length at
 262 most $2d + 1$ by Lemma 8. We can find a largest induced cycle C of length at most $2d + 1$
 263 in $O(n^{2d+1})$ time. Let $|V(C)| = p$. We write $N_0 = V(C) = \{x_1, x_2, \dots, x_p\}$ and for $i \geq 1$,
 264 $N_i = N(N_{i-1}) \setminus N_{i-2}$. So the sets N_i partition $V(G)$, and the distance of a vertex $u \in N_i$ to
 265 N_0 is i .

266 **Case 1.** $4 \leq p \leq 2d + 1$.

267 This case is illustrated in Figure 3. We consider every possible 3-colouring of C . Let c
 268 be such a 3-colouring. Every vertex with two differently coloured neighbours can only be
 269 coloured with one remaining colour. We assign this unique colour to such a vertex and apply
 270 this rule as long as possible. This takes polynomial time. The remaining vertices have a list
 271 of admissible colours that either consists of two or three colours, and vertices in the latter
 272 case belong to $V(G) \setminus (N_0 \cup N_1)$ (as $N(N_0) = N_1$).

273 If $N_2 = \emptyset$, then $V(G) = N_0 \cup N_1$. Then, we obtained an instance of 2-LIST COLOURING,
 274 which we can solve in linear time due to Theorem 6. Now assume that $N_2 \neq \emptyset$. Let $z \in N_2$.
 275 Then z has a neighbour $y \in N_1$, which in turn has a neighbour $x \in N_0$. If y is adjacent to
 276 neither neighbour of x on N_0 , then z, y, x and these two neighbours induce a chair in G ,
 277 a contradiction. Hence, y must be adjacent to at least one neighbour of x on N_0 , meaning
 278 that y must have received a colour by our algorithm. Consequently, z must have a list of
 279 admissible colours of size at most 2.

280 From the above we deduce that every vertex in N_2 has only two available colours in its list.
 281 We now consider the vertices of N_3 . Let $z' \in N_3$. Then z' has a neighbour $z \in N_2$, which in
 282 turn has a neighbour $y \in N_1$, which in turn has a neighbour $x \in N_0$, say $x = x_1$. If y has
 283 two non-adjacent neighbours in N_0 , then z', z, y and these two non-adjacent neighbours of y
 284 induce a chair in G , a contradiction. Combined with the fact deduced above, we conclude
 285 that y must have exactly two neighbours in N_0 and these two neighbours must be adjacent,
 286 say x_2 is the other neighbour of y in N_0 .

287 Suppose x_1 and x_2 are both adjacent to a vertex $y' \in N_1 \setminus \{y\}$ that is adjacent to a vertex
 288 in N_2 that has a neighbour in N_3 . Then, just as in the case of vertex y , the two vertices
 289 x_1 and x_2 are the only two neighbours of y' in N_0 . If y and y' are not adjacent, this means
 290 that x_2, x_3, x_4, y, y' induce a chair in G , a contradiction. Hence y and y' must be adjacent.
 291 However, then x_1, x_2, y, y' form a K_4 , a contradiction. This means that every pair of adjacent
 292 vertices of N_0 can have at most one common neighbour in N_1 that is adjacent to a vertex in
 293 N_2 with a neighbour in N_3 . We already deduced that every vertex of N_1 with a “descendant”
 294 in N_3 has exactly two neighbours in N_0 , which are adjacent. Hence, we conclude that the
 295 number of such vertices of N_1 is at most p .

296 We now observe that for $i \geq 2$, every vertex in N_i has at most two neighbours in N_{i+1} .
 297 This can be seen as follows. If $v \in N_i$ has two non-adjacent neighbours w_1, w_2 in N_{i+1} , then
 298 we pick a neighbour u of v in N_{i-1} , which has a neighbour t in N_{i-2} . Then v, u, t, w_1, w_2
 299 induce a chair in G , a contradiction. Hence, the neighbourhood of every vertex in N_i in
 300 N_{i+1} is a clique, which must have size at most 2 due to the K_4 -freeness of G . As the number
 301 of vertices in N_1 with a “descendant” in N_3 is at most p , this means that there are at most
 302 $2^{i-1}p$ vertices in N_i with a neighbour in N_{i+1} . Therefore the total number of vertices not
 303 belonging to any of the sets N_0, N_1 or N_2 is at most $\sum_{i=3}^d 2^{i-1}p$.

304 This means the total number of vertices not belonging to N_1 or N_2 is at most $\beta(d) =$
 305 $\sum_{i=3}^d 2^{i-1}p + p \leq \sum_{i=3}^d 2^{i-1}(2d + 1) + 2d + 1$. Let T_c be this set. We consider every possible
 306 3-colouring of $G[T_c]$. As we already deduced that the vertices in $N_1 \cup N_2$ have a list of size
 307 at most 2, for each case we obtain an instance of 2-LIST COLOURING, which we can solve in
 308 linear time due to Theorem 6. As the total number of instances we need to consider is at
 309 most $3^p \times 3^{\beta(d)} \leq 3^{2d+1} \times 3^{\beta(d)}$, our algorithm runs in polynomial time.

310 **Case 2.** $p = 3$.

311 As p was the size of a largest induced cycle of length at most $2d + 1$ and $2d + 1 \geq 5$, we find
 312 that G is C_4 -free. As G is K_4 -free, each vertex of N_1 is adjacent to at most two vertices of
 313 N_0 . If a vertex $x \in N_0$ has two independent private neighbours u and v in N_1 with respect

314 to N_0 , then every neighbour w of u in N_2 must also be a neighbour of v and vice versa, since
 315 G is chair-free. However, this is not possible, as x, u, w, v induce a C_4 . We conclude that
 316 u and v must be adjacent. Therefore, as G is K_4 -free, every vertex of N_0 has at most two
 317 private neighbours in N_1 , with respect to N_0 , that have a neighbour in N_2 .

318 By the same arguments as above we deduce that every two vertices of N_0 have at most
 319 one common neighbour in N_1 that is adjacent to a vertex in N_2 . Combined with the above,
 320 we find that there at most $6 + 3 = 9$ vertices in N_1 that have a neighbour in N_2 . If a vertex
 321 in N_1 has two independent neighbours in N_2 , then G contains an induced chair, which is
 322 not possible. Hence the neighbourhood of a vertex in N_1 in N_2 is a clique, which has size
 323 at most 2 due to the K_4 -freeness of G . We conclude that $|N_2| \leq 9 \times 2 = 18$. Similarly,
 324 every vertex in N_i for $i \geq 3$ has at most two neighbours in N_{i+1} . Therefore the number of
 325 vertices in N_i for $i \geq 3$ is at most $18 \times 2^{i-2}$. This means that the total number of vertices
 326 outside $N_0 \cup N_1 \cup N_2$ is at most $\beta(d) = \sum_{i=3}^d 18 \times 2^{i-2}$. Let T be this set. We consider every
 327 possible 3-colouring of $G[T]$ and every possible 3-colouring of C . For each case we obtain an
 328 instance of 2-LIST COLOURING, which we can solve in linear time due to Theorem 6. As the
 329 total number of instances we need to consider is at most $3^d \times 3^{\beta(d)}$, our algorithm runs in
 330 polynomial time.

331 **2.** Let G be a $K_{1,r}^2$ -free graph of diameter at most 2. We first check in $O(n^4)$ time if G is
 332 K_4 -free. If not, then G is not 3-colourable. We then check in $O(n^5)$ time if G has an induced
 333 C_5 . If G is C_5 -free, then we use Theorem 11. From now on, suppose that G is K_4 -free and
 334 that G contains an induced cycle C of length 5, say on vertices x_1, \dots, x_5 in that order. We
 335 write $N_0 = V(C) = \{x_1, \dots, x_5\}$, $N_1 = N(V(C))$ and $N_2 = V(G) \setminus (N_0 \cup N_1)$.

336 Let N'_2 be the set of vertices in N_2 that are adjacent to some vertex in N_1 that is a
 337 private neighbour of some vertex in N_0 with respect to N_0 . As G is K_4 -free, the private
 338 neighbourhood $P(x_i)$ of each vertex $x_i \in N_0$ with respect to N_0 does not contain a clique of
 339 size 3. Moreover, if $P(x_i)$ contains an independent set I of size $r - 1$ for some $i \in \{1, \dots, 5\}$,
 340 then $I \cup \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$ induces a $K_{1,r}^2$, which is not possible. Now let $v \in P(x_i)$
 341 for some $i \in \{1, \dots, 5\}$, say $i = 1$. As G is K_4 -free, the set $N(v) \cap N_2$ does not contain a
 342 clique of size 3. Moreover, if $N(v) \cap N_2$ contains an independent set I' of size $r - 1$, then
 343 $I' \cup \{v, x_1, x_2, x_3\}$ induces a $K_{1,r}^2$, which is not possible. Hence, $|N(v) \cap N_2| \leq R(3, r - 1)$
 344 by Ramsey's Theorem. We conclude that $|N'_2| \leq 5R(3, r - 1)^2$.

345 We now consider all possible 3-colourings of C . Let c be such a 3-colouring. We assume
 346 without loss of generality that $c(x_1) = c(x_3) = 1$, $c(x_2) = c(x_4) = 2$ and $c(x_5) = 3$. Moreover,
 347 every vertex that has two differently coloured neighbours can only be coloured with one
 348 remaining colour. We assign this unique colour to such a vertex and apply this rule as long
 349 as possible. This takes polynomial time. The remaining vertices have a list of admissible
 350 colours that either consists of two or three colours, and vertices in the latter case must belong
 351 to N_2 (as $N(N_0) = N_1$).

352 Let T_c be the set of vertices in N_2 that still have a list of size 3. We will prove that
 353 $T_c \subseteq N'_2$. Let $u \in T_c$. As G has diameter 2, we find that u has a neighbour v adjacent to x_5 .
 354 Then v cannot be adjacent to any of x_1, \dots, x_4 , as otherwise v would have a unique colour
 355 and u would not be in T_c . Hence, v is a private neighbour of x_5 with respect to N_0 . We
 356 conclude that all vertices in T_c belong to N'_2 , which implies that $|T_c| \leq |N'_2| \leq 5R(3, r - 1)^2$.

357 We now consider every possible 3-colouring of $G[T_c]$. Then all uncoloured vertices have a
 358 list of size at most 2. In other words, we created an instance of 2-LIST COLOURING, which
 359 we solve in linear time using Theorem 6. As the number of 3-colourings of C is at most 3^5
 360 and for each 3-colouring c of C the number of 3-colourings of $G[T_c]$ is at most $3^{5R(3, r - 1)^2}$,
 361 the total running time of our algorithm is polynomial.

362 **3.** We consider the standard reduction from the NP-complete problem NAE 3-SAT [29],
 363 where each variable appears in at most three clauses and each literal appears in at most two.
 364 Given a CNF formula ϕ , we construct the graph G as follows:

- 365 ■ Add a vertex v_{x_i} for each literal x_i .
- 366 ■ Add an edge between each literal and its negation.
- 367 ■ Add a vertex z adjacent to every literal vertex.
- 368 ■ For each clause C_i add a triangle T_i with vertices $c_{i_1}, c_{i_2}, c_{i_3}$.
- 369 ■ Fix an arbitrary order of the literals of C_i , $x_{i_1}, x_{i_2}, x_{i_3}$ and add an edge $x_{i_j}c_{i_j}$.

370 Given a 3-colouring of G , assume z is assigned colour 1. Then each literal vertex is
 371 assigned either colour 2 or colour 3. If, for some clause C_i , the vertices x_{i_1}, x_{i_2} and x_{i_3} are
 372 all assigned the same colour, then T_i cannot be coloured. Therefore, if we set literals whose
 373 vertices are coloured with colour 2 to be true and those coloured with colour 3 to be false,
 374 each clause must contain at least one true literal and at least one false literal.

375 If ϕ is satisfiable then we can colour vertex z with colour 1, each true literal with colour 2
 376 and each false literal with colour 3. Then, since each clause has at least one true literal and
 377 at least one false literal, each triangle has neighbours in two different colours. This implies
 378 that each triangle is 3-colourable. Therefore G is 3-colourable if and only if ϕ is satisfiable.

379 We next show that G has diameter at most 4. First note that any literal vertex is adjacent
 380 to z and any clause vertex is adjacent to some literal vertex so any vertex is at distance at
 381 most 2 from z . Therefore any two vertices are at distance at most 4.

382 Finally we show that G is $K_{1,4}^3$ -free. Any literal vertex has degree at most 4 since it
 383 appears in at most two clauses. However it has at most 3 independent neighbours since its
 384 negation is adjacent to z . Each clause vertex has at most 3 neighbours so the only vertex
 385 with four independent neighbours is d . The longest induced path including z has length
 386 at most 4 since any such path contains at most one literal and at most two vertices of any
 387 triangle. Therefore G is $K_{1,4}^3$ -free.

388 **4.** This follows from Theorem 3. Let $k^* \geq 3$. We take a claw-free graph G and add a
 389 dominating vertex to it. The new graph G' has diameter at most 2 and is $K_{1,3}^1$ -free. Let
 390 $k = k^* + 1 \geq 4$. Then G is k^* -colourable if and only if G' is k -colourable. ◀

391 Subdividing two edges of the claw yields another interesting case, namely where $H = S_{1,2,2}$.
 392 For $k \geq 4$, Theorem 12 tells us that k -COLOURING is NP-complete for $S_{1,2,2}$ -free graphs of
 393 diameter 2. For $k = 3$, we could only prove polynomial-time solvability if $d = 2$.

394 ► **Theorem 13.** 3-COLOURING can be solved in polynomial time for $S_{1,2,2}$ -free graphs of
 395 diameter at most 2.

396 **Proof.** Let G be an $S_{1,2,2}$ -free graph of diameter at most 2. We first check in $O(n^5)$
 397 time if G has an induced C_5 . If G is C_5 -free, then we use Theorem 11. Suppose G
 398 contains an induced cycle C of length 5, say on vertices x_1, \dots, x_5 in that order. We write
 399 $N_0 = V(C) = \{x_1, \dots, x_5\}$, $N_1 = N(V(C))$ and $N_2 = V(G) \setminus (N_0 \cup N_1)$. As G has diameter 2,
 400 for every $i \in \{1, 2, 3\}$, every vertex in N_2 has a neighbour in N_1 that is adjacent to x_i .

401 We let T consist of all vertices of N_2 that have a neighbour in N_1 that is adjacent to two
 402 adjacent vertices of N_0 . So the colour of any vertex of T will be fixed in any 3-colouring after
 403 colouring the five vertices of N_0 . We claim that $N_2 = T$. In order to see this, let $u \in N_2$. As
 404 G has diameter 2, we find that u must have a neighbour $v \in N_1$ adjacent to a vertex of N_0 ,
 405 say x_1 . Then v is not adjacent to x_5 or x_2 . If v is not adjacent to x_3 either, then the vertices
 406 x_1, x_5, x_2, x_3, v, u induce a $S_{1,2,2}$ with center x_1 , a contradiction. So v must be adjacent to

407 x_3 , meaning v is not adjacent to x_4 . However, now x_3, x_2, x_4, x_5, v, u induce a $S_{1,2,2}$ with
 408 center x_3 , another contradiction.

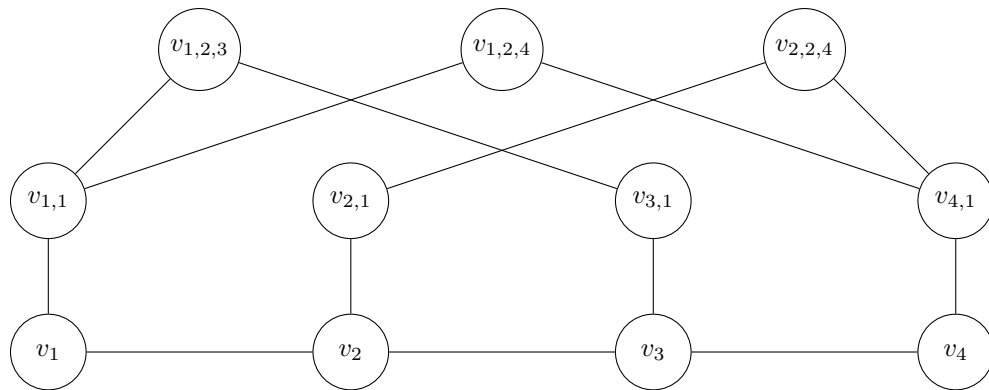
409 We now “guess” the 3-colouring of C by considering all 3^5 possibilities if necessary. We
 410 then proceed as in the proof of Theorem 11. That is, we observe that every vertex of N_1
 411 can only be coloured with two possible colours and that after propagation, every uncoloured
 412 vertex of N_2 can only be coloured with two possible colours as well (as $T = N_2$). Then it
 413 remains to solve an instance of 2-LIST COLOURING, which takes linear time by Theorem 6. As
 414 we need to do this at most 3^5 times, the total running time of our algorithm is polynomial. ◀

415 4 Graphs of Bounded Diameter and Girth

416 In this section we will examine the trade-offs for k -COLOURING between diameter and girth.
 417 Recall that Mertzios and Spirakis [24] proved that 3-COLOURING is NP-complete for graphs
 418 of diameter 3 and girth 4 (Theorem 5). We extend their result in our next theorem, partially
 419 displayed in Figure 2. This theorem shows that there is still a large gap for which we do not
 420 know the computational complexity of 3-COLOURING for graphs of diameter d and girth g .

421 ► **Theorem 14.** *Let d, g, k be three integers with $d \geq 2$, $g \geq 3$ and $k \geq 3$. Then k -COLOURING
 422 for graphs of diameter at most d and girth at least g is*

- 423 1. *polynomial-time solvable if $g \geq 2d + 1$*
- 424 2. *NP-complete if $d = 3$ and $g \leq 4$ and $k = 3$*
- 425 3. *NP-complete if $4p \leq d \leq 4p + 3$ and $g \leq 4p + 2$ for some integer $p \geq 1$ and $k = 3$.*



■ **Figure 4** An example of a graph G' , constructed in the proof of Theorem 14(3), for $p = 1$.

426
 427 **Proof.** 1. This case follows from Theorem 7. 2. This case is Theorem 5 (proven in [24]).

428 3. We reduce 3-COLOURING for graphs of girth at least $8p - 3$, which is NP-complete by
 429 Theorem 4, to 3-COLOURING for graphs of diameter at most $4p$ and girth at least $4p + 2$.
 430 Construct the graph G' as follows (see Figure 4 for an example):

- 431 ■ label the vertices of G v_1 to v_n ;
- 432 ■ for each vertex of G , add a new neighbour $v_{i,1}$;
- 433 ■ for every two vertices v_i and v_j such that $\text{dist}(v_i, v_j) > l = 2p - 1$ add new vertices to
 434 form the path $v_{i,1}v_{i,2,j} \dots v_{i,p+1,j}v_{j,p,i} \dots v_{j,1}$.

First we show that G' has diameter at most $4p$. For any two vertices v_i and v_j of G either $\text{dist}(v_i, v_j) \leq l$ or we have the path $v_{i,1}v_{i,2,j}\dots v_{i,p+1,j}v_{j,p,i}\dots v_{j,1}$ and $\text{dist}(v_i, v_j) \leq 2p + 2$. Similarly, $\text{dist}(v_i, v_{j,1}) \leq 2p + 1$ and $\text{dist}(v_{i,1}, v_{j,1}) \leq 2p + 1$. Now consider two vertices $v_{a,r,b}$ and $v_{c,q,d}$ for $2 \leq r \leq p + 1$, $2 \leq q \leq p + 1$. If $\text{dist}(v_a, v_c) \leq l$ then $\text{dist}(v_{a,r,b}, v_{c,q,d}) \leq r + q + l \leq (p + 1) + (p + 1) + (2p - 1) \leq 4p + 1$. Otherwise we have the path $v_{a,r,b}\dots v_{a,1}v_{a,2,c}\dots v_{a,p+1,c}v_{c,p,a}\dots v_{c,1}v_{c,2,d}\dots v_{c,q,d}$. This gives $\text{dist}(v_{a,r,b}, v_{c,q,d}) \leq (r - 1) + p + p + (q - 1) \leq 4p$. In fact, if $\text{dist}(v_{a,r,b}, v_{c,q,d}) = 4p + 1$, then we must have $r = q = p + 1$ and $\text{dist}(v_a, v_c) = \text{dist}(v_a, v_d) = \text{dist}(v_b, v_c) = \text{dist}(v_b, v_d) = 2p - 1$. In this case we have two paths of length at most $4p - 2$ between v_a and v_b , one containing v_c and the other containing v_d . These paths must be distinct since the existence of the vertex $v_{c,p+1,d}$ implies that $\text{dist}(v_c, v_d) > 2p - 1$. Therefore we have a cycle in G of length at most $8p - 4$ which contradicts the assumption that G has girth at least $8p - 3$. This implies that the diameter of G' is at most $4p$.

Since G has girth at least $8p - 3$, every cycle in G' of length less than $4p + 2$ must contain at least one vertex of $V(G') \setminus V(G)$. Since all the vertices of $V(G') \setminus V(G)$ except the vertices $v_{i,1}$ have degree 2, any such cycle C must contain the path $v_{i,1}\dots v_{i,p+1,j}\dots v_j$ for some v_i, v_j at distance greater than l . This path has length $2p + 1$. If C contains $v_{i,2,m}$ for some m different from j then it contains the path $v_{i,2,m}\dots v_{m,1}$ and has length at least $4p + 2$. Similarly, this is the case if C contains $v_{j,2,m}$ for m different from i . Otherwise C contains v_i and v_j which are at distance at least l and has length at least $(2p + 1) + 2 + (2p - 1) = 4p + 2$.

Finally, we show that G is 3-colourable if and only if G' is 3-colourable. The latter holds if and only if the subgraph G'' of G' induced by $V(G) \cup \{v_{i,1} \mid 1 \leq i \leq n\}$ is 3-colourable, since every other vertex of G' has degree 2. The graph G is 3-colourable if and only if G'' is 3-colourable, since G is an induced subgraph of G'' and each vertex of $V(G'') \setminus V(G)$ has degree 1. Therefore, G is 3-colourable if and only if G' is 3-colourable. ◀

5 Conclusions

We proved a number of new results for COLOURING and k -COLOURING for polyad-free graphs of bounded diameter and for graphs of bounded diameter and girth. In particular we identified and narrowed a number of complexity gaps. This leads us to some natural open problems. Our first two open problems follow from Theorem 10. The third open problem comes from Theorem 12; note that $K_{1,3}^2 = S_{1,1,3}$. Our fourth open problem stems from Theorem 13. Recall that determining the complexity of 3-COLOURING for graphs of diameter 2 is still wide open. This question is covered by the fifth open problem.

▷ Open Problem 1. Does there exist an integer d such that COLOURING is NP-complete for $K_{1,3}$ -free graphs of diameter d ?

▷ Open Problem 2. What is the complexity of COLOURING for C_3 -free graphs of diameter 2, or equivalently, graphs of diameter 2 and girth 4?

▷ Open Problem 3. What are the complexities of 3-COLOURING for $K_{1,4}^1$ -free graphs of diameter 3 and for $K_{1,3}^2$ -free graphs of diameter 3?

▷ Open Problem 4. Do there exist integers d, h, i, j such that 3-COLOURING is NP-complete for $S_{h,i,j}$ -free graphs of diameter d ?

▷ Open Problem 5. What is the complexity of the open cases in Figure 2 and in particular of 3-COLOURING for graphs of diameter 2 and for graphs of diameter 2 and girth 4?

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